The Influence of the Energy Dissipation and of the Geometry on Toroidal Resonators with a Conducting Separating Wall

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Abstract—**An exact solution of the Maxwell equations for the stationary electromagnetic waves in a toroidal resonator with a separating wall is obtained. The structure of the fields in the resonator and in the metallic toroidal wall is described analytically. The dispersion relation is formulated and the eigenfrequencies, the damping rate and the Q factor of the resonator are calculated.**

I. Introduction

THE FUNCTION

$$f(\rho,\theta,\phi)=\frac{AZ_\nu(k\rho)}{\sqrt{1-\rho\cos\theta}}\cos(\nu\theta-\alpha)\sin\frac{\phi}{2} \tag{1}$$

is an exact solution of the scalar Helmholtz equation

$$\frac{\partial}{\partial\rho}\rho(1-\rho\cos\theta)\frac{\partial f}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\theta}(1-\rho\cos\theta)\frac{\partial f}{\partial\theta}+\frac{\rho}{1-\rho\cos\theta}\frac{\partial^2 f}{\partial\phi^2}+k^2\rho(1-\rho\cos\theta)f=0. \tag{2}$$

Here we use the quasi-toroidal coordinates ρ,θ,ϕ, which are related to the Cartesian coordinates by the equations $x=R(1-\rho\cos\theta)\cos\phi$, $y=R(1-\rho\cos\theta)\sin\phi$, and $z=R\rho\sin\theta$ with R as major radius of the torus. Z_ν is a cylindrical function of the order ν. A, k, and α are arbitrary constants.

Solution (1) can be used to describe physical phenomena, which can be reduced to the solution of the scalar Helmholtz equation. Such a phenomenon is, e.g., the stationary electromagnetic oscillation in a torus. In this case the introduction of the Hertz vector in a special way [1] allows the reduction of the whole mathematical problem to the solution of (2). But the dependence on $\sin\phi/2$ implies a periodicity of the field with period 4π, which can be realized in practice with a conducting separating wall (corresponding to a $\phi=$const plane.)

We tried to study the stationary electromagnetic waves in a torus with a separating wall in a former paper [2]. Here we construct the complete system of independent Hertz vectors, which can be obtained from the generating function $f(\rho,\theta,\phi)$ given in (1). Three different classes of fields were obtained, but these satisfied only the condition

$$B_\rho=0 \tag{3}$$

at the conducting toroidal wall ($\rho=\rho_0$). For a wall with conductivity $\sigma=\infty$ conditions

$$E_\phi=E_\theta=0 \tag{4}$$

must also be satisfied. To satisfy these conditions we must superpose the fields described in [2]. Studying different possibilities of superposition we really got only one, which satisfies all boundary conditions for the fields, and the field components satisfy exactly the Maxwell equations for stationary electromagnetic waves in the inner of the torus. We give in the first part of the paper the solution resulting from the mentioned superposition and analyze the structure of the electromagnetic field in the torus. In the second part we study the influence of finite conductivity of the walls, where we determine the structure of the field in the toroidal wall, the damping rate and the Q factor of the resonator.

II. The Electromagnetic Field of the Resonators, if the Conductivity of the Walls is $\sigma=\infty$

Through a superposition of the fields resulting from (1) and described in [2] we get the electromagnetic field

$$\begin{aligned}E_\rho&=0\\ E_\theta&=\frac{AJ_1(k\rho)}{\sqrt{1-\rho\cos\theta}}\cos\frac{\phi}{2}\\ E_\phi&=-\frac{AJ_1(k\rho)}{\sqrt{1-\rho\cos\theta}}\sin\theta\sin\frac{\phi}{2}\end{aligned} \tag{5}$$

$$\begin{aligned}B_\rho&=-i\frac{cAJ_1(k\rho)}{2\omega R\rho\sqrt{(1-\rho\cos\theta)^3}}(2-3\rho\cos\theta)\cos\theta\sin\frac{\phi}{2}\\ B_\theta&=i\frac{cA}{2\omega R\sqrt{1-\rho\cos\theta}}\left(2kJ_0-\frac{(2-\rho\cos\theta)J_1}{\rho(1-\rho\cos\theta)}\right)\sin\theta\sin\frac{\phi}{2}\\ B_\phi&=i\frac{cA}{2\omega R\sqrt{1-\rho\cos\theta}}\left(2kJ_0+\frac{J_1\cos\theta}{1-\rho\cos\theta}\right)\cos\frac{\phi}{2}.\end{aligned} \tag{6}$$

It is easy to verify by a direct substitution of (5) and (6) in the Maxwell equations for stationary electromagnetic

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waves

$$\operatorname{curl} \boldsymbol{E} = -i\frac{\omega}{c}\boldsymbol{B}$$

$$\operatorname{curl} \boldsymbol{B} = i\frac{\omega\epsilon\mu}{c}\boldsymbol{E}$$

$$\operatorname{div} \epsilon \boldsymbol{E} = 0$$

$$\operatorname{div} \boldsymbol{B} = 0 \tag{7}$$

that they are satisfied. Here we mention that in (1), (2), (5), and (6) we used the notation

$$k^2 = \frac{\epsilon\mu\omega^2 R^2}{c^2} \tag{8}$$

with ω as angular frequency. Curl $\boldsymbol{E}$ and div$\boldsymbol{E}$ in quasi-toroidal coordinates are

$$(\operatorname{curl} \boldsymbol{E})_\rho = \frac{1}{R\rho(1-\rho\cos\theta)}\left\{\frac{\partial}{\partial\theta}(1-\rho\cos\theta)E_\phi - \frac{\partial}{\partial\phi}(\rho E_\theta)\right\}$$

$$(\operatorname{curl} \boldsymbol{E})_\theta = \frac{1}{R(1-\rho\cos\theta)}\left\{\frac{\partial E_\rho}{\partial\phi} - \frac{\partial}{\partial\rho}(1-\rho\cos\theta)E_\phi\right\}$$

$$(\operatorname{curl} \boldsymbol{E})_\phi = \frac{1}{R\rho}\left\{\frac{\partial(\rho E_\theta)}{\partial\rho} - \frac{\partial E_\rho}{\partial\theta}\right\}$$

$$\operatorname{div} \boldsymbol{E} = \frac{1}{R\rho(1-\rho\cos\theta)}\left\{\frac{\partial(1-\rho\cos\theta)\rho E_\rho}{\partial\rho} + \frac{\partial(1-\rho\cos\theta)E_\theta}{\partial\theta} + \frac{\partial(\rho E_\phi)}{\partial\phi}\right\}. \tag{9}$$

In (5) and (6) we replaced Z_ν by the Bessel function J_ν (the Neumann function would lead to $|\boldsymbol{E}| = \infty$ and $|\boldsymbol{B}| = \infty$ for $\rho = 0$).

Using (5) we can integrate the equations for the electric field lines of the electromagnetic field. We get

$$\rho = \text{const} \quad \text{and} \quad \sqrt{1-\rho\cos\theta}\,\sin\frac{\phi}{2} = \text{const}. \tag{10}$$

Therefore the electric field lines are spatial curves which remain always on the same $\rho = \text{const}$ surface. Their projections on the $\theta = 0$ and $\theta = \pi$ planes are the parabolas plotted in Fig. 1. In Fig. 2 we plot them in a perspective view.

At the separating wall a discontinuity exists. This is the only surface where surface charges appear (on the toroidal wall E_ρ is always zero.) The charge distribution on the separating wall is described by the surface charge density

$$\sigma_s \sim \frac{J_1(k\rho)}{\sqrt{1-\rho\cos\theta}}\sin\theta. \tag{11}$$

As can be seen, the surface charge density has a dipolar structure, and the density oscillation on this surface is a dipolar oscillation perpendicular to the plane represented in Fig. 1. One part of the field lines are starting from the positive region of the separating wall, and remaining always on the $\rho = \text{const}$ torus they return to the negative region of the separating wall. The other part of the electric field lines are closed lines on the $\rho = \text{const}$ surfaces. The magnetic field has a more complicated three dimensional structure. The components of the field are given by (6). At

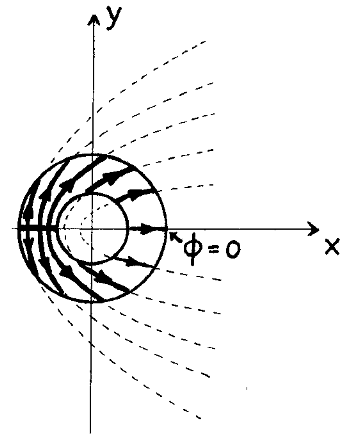

Fig. 1. Projections of the electric field lines.

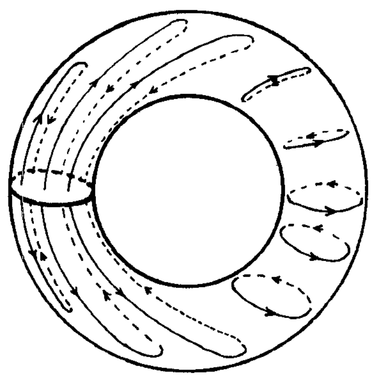

Fig. 2. Electric field lines.

the toroidal wall ($\rho = \rho_0$) the magnetic field lines are described by the following equation:

$$\frac{(1-\cos\theta)^{\rho_0/1-\rho_0}}{(1+\cos\theta)^{\rho_0/1+\rho_0}(1-\rho_0\cos\theta)^{2\rho_0^2/1-\rho_0^2}}\cos^4\frac{\phi}{2} = C.$$

The magnetic field has a symmetrical structure against the $\theta = 0$, $\theta = \pi$ plane. The field lines do not traverse this plane.

III. The Eigenfrequencies of the Resonator with Very Good Conducting Wall

Solutions (5)–(6) satisfies the boundary conditions (3) and (4) if

$$J_1(k\rho_0) = 0. \tag{12}$$

Therefore we get the eigenfrequencies

$$\omega_m = \frac{c}{r_0}y_{1,m}. \tag{13}$$

Here $y_{1,m}$ are the roots of the Bessel function J_1 and r_0 is the minor radius of the torus.

We can compare formally these eigenfrequencies with the eigenfrequencies of a cylinder [3]:

$$\omega_{lmn}^E = c\sqrt{\left(\frac{y_{lm}}{r_0}\right)^2 + \left(\frac{n\pi}{h}\right)^2}, \quad \text{for the } E \text{ modes}$$

$$\omega_{lmn}^H = c\sqrt{\left(\frac{y'_{lm}}{r_0}\right)^2 + \left(\frac{n\pi}{h}\right)^2}, \quad \text{for the } H \text{ modes}$$

y_{lm} and y'_{lm} are the roots of $J_l(y)=0$ and $dJ_l(y)/dy=0$, respectively. Equation (13) results formally from these equations for $h=\infty$ if we take $l=1$ by the E modes or $l=0$ by the H modes. But at the cylinder we have three parameters (n, m, l) and by our eigenfrequencies for the torus we have only the parameter m. This difference is related to the geometrical properties of the torus. If we compare the torus with the cylinder, we can see that the second curvature of the torus reduces the number of the symmetry planes. In the cylinder all planes $\theta=$const contain the axis of the cylinder and are symmetry planes. In the torus only the $\theta=0$, $\theta=\pi$ plane is a symmetry plane containing the toroidal axis. This has as consequence, that we do not have in our case exact solutions described by the Bessel function J_ν with arbitrary ν (as in the case of the cylinder), but only by J_1. This solution is given by (5)–(6). The other solutions could be eventually constructed by a series development against the functions resulting from (1) for arbitrary ν. Here we restrict our analysis to the modes described by (5) and (6). At the same time we would like to emphasize also that the dispersion relation (13) has the same form as the equivalent relation of paper [2] with the sole restriction that it can be applied only for the roots of J_1.

The fundamental frequency resulting from (13) is

$$\omega_1=3.832\frac{c}{r_0}. \tag{14}$$

IV. The Influence of Finite Conductivity

A. The Q Factor of the Resonator

Here we suppose that the skin depth (δ) for the electromagnetic waves in the metallic wall is much less than the dimensions of the torus (minor radius). In this case we can determine the power losses in the cavity in the first approximation using the well known relation [3]

$$P_{\text{loss}}=\frac{c^2}{32\pi^2\sigma\mu^2\delta}\int_S[\boldsymbol{n}\times\boldsymbol{B}]^2\,dS \tag{15}$$

where μ is the magnetic permeability in the resonator, σ the electric conductivity of the wall, δ the skin depth and $\boldsymbol{n}$ the normal vector to the metallic surface. The integration is extended over the whole surface surrounding the resonant cavity. The surface of the conducting separating wall is about $\rho_0/4\pi$ ($\ll 1$) times the surface of the torus. Therefore the dominant effect is the energy dissipation in the toroidal wall. Taking this into account we get from (6) and (15)

$$P_{\text{loss}}\approx\frac{3\epsilon\rho_0c^2A^2R^2}{32\sigma\mu\delta}J_0^2(k\rho_0). \tag{16}$$

For the electromagnetic energy

$$U=\frac{1}{16\pi}\int_V(\boldsymbol{EE}^*+\boldsymbol{BB}^*)\,dV$$

stored in the cavity from (5) and (6) results

$$U=\tfrac{3}{16}\pi\rho_0^2A^2R^3J_2^2(k\rho_0). \tag{17}$$

Taking into account, that for the roots of J_1 we have $J_0(y_{1,m})=-J_2(y_{1,m})$ and using the definition of the Q factor, we get

$$Q=\frac{\mu}{\mu_c}\frac{\rho_0R}{\delta}. \tag{18}$$

Here μ_c is the magnetic permeability of the conducting wall. So for the geometrical factor of the toroidal resonator with a separating wall we get ≈ 2, therefore the same order of magnitude as for the TM modes in cylindrical cavities.

B. The Structure of the Field in the Toroidal Wall with Finite Conductivity and the Corresponding Dispersion Relation

Here we suppose that only the separating wall is very thin and of very large conductivity. We suppose that the toroidal wall has a finite conductivity satisfying the condition $\sigma\gg\epsilon\omega/4\pi$. In this case we can solve the Maxwell equations separately for the inside of the cavity and for the conducting wall, and after that we can impose the conditions which must be satisfied at the interface of the two media. The solution for the inside of the toroidal cavity is identical with (5) and (6). For the conducting toroidal wall we have to solve the Maxwell equations

$$\begin{aligned}\operatorname{curl}\boldsymbol{E}&=-i\frac{\omega}{c}\boldsymbol{B}\\ \operatorname{curl}\boldsymbol{B}&=\frac{4\pi\sigma\mu_c}{c}\boldsymbol{E}\\ \operatorname{div}\epsilon\boldsymbol{E}&=0\\ \operatorname{div}\boldsymbol{B}&=0.\end{aligned} \tag{19}$$

The exact solution of this system, which can be coupled at the interface with (5) and (6) is

$$E_\rho=0$$

$$E_\theta=\alpha e^{i[\eta_1(\kappa\rho_0)-\eta_1(\kappa\rho)]}\frac{h_1(\kappa\rho)\cos\phi/2}{h_1(\kappa\rho_0)\sqrt{1-\rho\cos\theta}}$$

$$E_\phi=-\alpha e^{i[\eta_1(\kappa\rho_0)-\eta_1(\kappa\rho)]}\frac{h_1(\kappa\rho)\sin\theta\sin\phi/2}{h_1(\kappa\rho_0)\sqrt{1-\rho\cos\theta}}$$

$$B_\rho=-i\frac{\alpha c}{2\omega R}e^{i[\eta_1(\kappa\rho_0)-\eta_1(\kappa\rho)]}\frac{h_1(\kappa\rho)(2-3\rho\cos\theta)}{\rho h_1(\kappa\rho_0)\sqrt{1-\rho\cos\theta}}\cdot\cos\theta\sin\frac{\phi}{2}$$

$$B_\theta=-i\frac{\alpha c}{2\omega R}e^{i[\eta_1(\kappa\rho_0)-\eta_1(\kappa\rho)]}\frac{h_1(\kappa\rho)(2-\rho\cos\theta)}{\rho h_1(\kappa\rho_0)\sqrt{(1-\rho\cos\theta)^3}}\cdot\sin\theta\sin\frac{\phi}{2}$$

$$+i\frac{\alpha c\kappa}{\omega R}e^{i[\eta_1(\kappa\rho_0)-\eta_0(\kappa\rho)-\pi/4]}\frac{h_0(\kappa\rho)\sin\theta\sin\frac{\phi}{2}}{h_1(\kappa\rho_0)\sqrt{1-\rho\cos\theta}}$$

$$B_\phi = i\frac{\alpha c}{2\omega R} e^{i[\eta_1(\kappa\rho_0)-\eta_1(\kappa\rho)]} \frac{h_1(\kappa\rho)\cos\theta\cos\frac{\phi}{2}}{h_1(\kappa\rho_0)\sqrt{(1-\rho\cos\theta)^3}} + i\frac{\alpha c\kappa}{\omega R} e^{i[\eta_1(\kappa\rho_0)-\eta_0(\kappa\rho)-\pi/4]} \frac{h_0(\kappa\rho)\cos\frac{\phi}{2}}{h_1(\kappa\rho_0)\sqrt{1-\rho\cos\theta}}.$$

$$\kappa = R/c\,(4\pi\sigma\omega\mu_c)^{1/2} \tag{20}$$

Here the functions h_0, h_1, η_0, η_1 are related to the Hankel functions $H_\nu^{(1)}$ through the relations

$$h_0(z)e^{-i\eta_0(z)} = -H_0^{(1)}(ze^{i3\pi/4})$$

$$h_1(z)e^{-i\eta_1(z)} = H_1^{(1)}(ze^{i3\pi/4}).$$

The functions h_0, h_1, η_0, and η_1 are tabulated in [4]. As can be seen, the amplitude of the electromagnetic field in the metallic wall is determined by h_1 and h_0. Both are decreasing exponentially for large $\kappa\rho$. η_1 and η_0 determine only the phase deviation.

Equations (5), (6), and (20) satisfy the conditions for the continuity of B_ρ, E_θ, and E_ϕ at the interface if

$$\alpha = AJ_1(k\rho_0). \tag{21}$$

If we take into account that the surface charges and the surface currents are distributed practically in the region with the dimensions of the skin depth, at the interface between the cavity and the conducting wall the quantities ϵE_ρ, $1/\mu B_\theta$ and $1/\mu B_\phi$ must be continuous. The first of these conditions is automatically satisfied because of $E_\rho \equiv 0$, and the other two conditions lead to the dispersion relation

$$\frac{J_1(k\rho_0)}{J_0(k\rho_0)} = \frac{\mu_c}{\mu}\sqrt{\frac{\epsilon\mu\omega}{4\pi\sigma\mu_0}}\,\frac{h_1(\kappa\rho_0)}{h_0(\kappa\rho_0)} e^{i[\eta_0(\kappa\rho_0)-\eta_1(\kappa\rho_0)+\pi/4]}. \tag{22}$$

Here we get the eigenfrequencies and the damping rates for various conductivities and permeabilities of the wall and various parameters of the cavity. For a very good conducting metallic wall we have

$$\frac{\epsilon\mu\omega}{4\pi\sigma\mu_c} \ll 1$$

and we get for the real part of ω the values given in (13). In this case $\operatorname{Im}\omega \ll \operatorname{Re}\omega$ and we can develop (22) in a series in $\operatorname{Im}\omega/\operatorname{Re}\omega$. We get for the damping rate

$$\gamma = \operatorname{Im}\omega = \frac{\mu_c}{2\mu}\,\frac{\omega\delta}{\rho_0 R}. \tag{23}$$

Here we would like to make a *remark*: using expressions (5) and (6) we can construct by superposition of $\boldsymbol{E}$ and $\boldsymbol{B}$ a vector, which is an exact solution of the equation

$$\operatorname{curl}\boldsymbol{F} = \boldsymbol{F}$$

corresponding in our case to a force-free magnetic field with periodicity 4π.

V. Conclusions

Using various superpositions of a finite number of basis functions (1) for the toroidal field of a resonator with a separating wall we could find only one field, which satisfies exactly all Maxwell equations and all boundary conditions. The structure of the stationary electromagnetic wave-field differs essentially from the field structures of the cylindrical cavity. The classification in E or H modes cannot be used for the symmetry of the described toroidal mode. The eigenfrequencies of the toroidal resonator with a separating wall coincide with the eigenfrequencies of the cylindrical E modes with $l=1$ and $h=\infty$. The geometrical factor in the expression of Q is of the same order as for the TM modes.

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